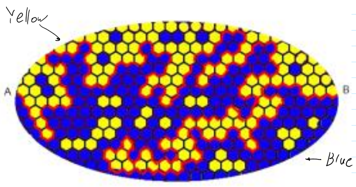
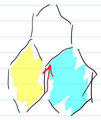
Now let us concentrate on convergence of Exploration Process for critical percolation.



$Perc_\delta$

Red path: Exploration Process



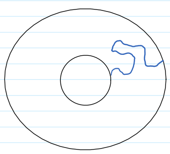
Turn left on Blue, right on Yellow; toss coin on new hexagon

Statement Critical Percolation Exploration Process satisfies KS condition.

Proof.
Follows from previously mentioned:

Lemma (Russo - Seymour - Welsh)
Let $A_z(r, 2r)$ - annulus centered at z, inner radius r, outer radius $2r$, $r \gg \delta$ ($r \geq 100\delta$). Then $\exists\, q > 0$.

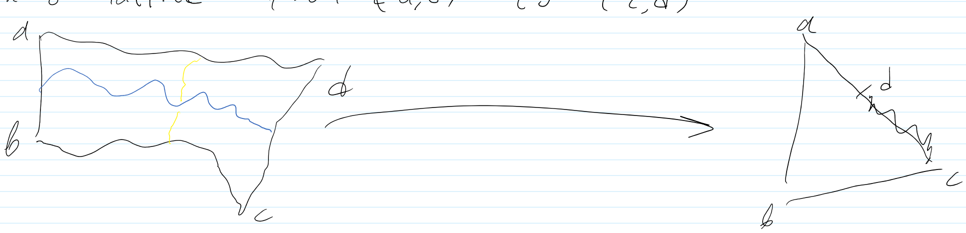
$1-q > P(\exists \text{ blue crossing of } A_z(r,2r)) > q$



So probability of unforced crossing of $A(z_0, r, R)$ is bounded by $1-q$ if $R > 100 r$. ∎

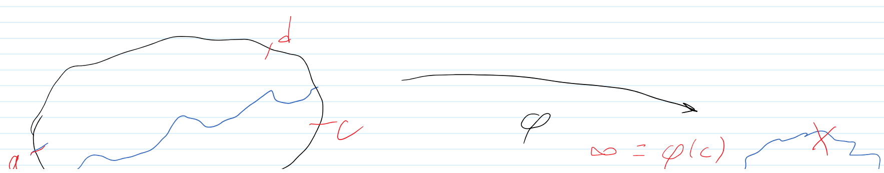
Cardy formula, revisited

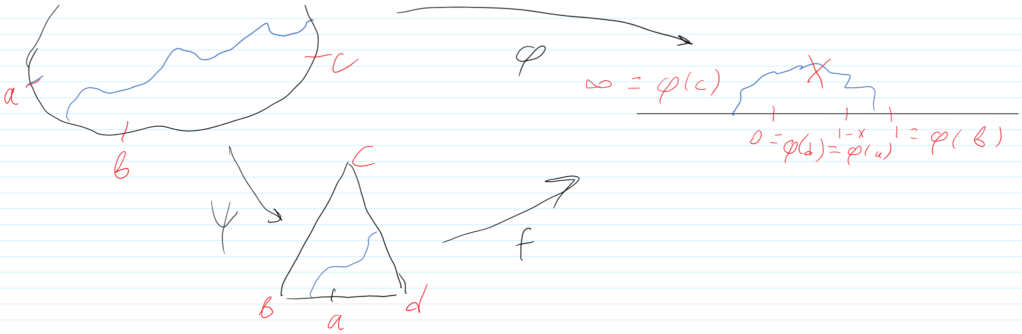
(Ω, a, b, c, d) - conformal rectangle
$C_\delta(\Omega, a, b, c, d)$ - probability of blue crossing on δ - lattice from (a,b) to (c,d)



Let $\varphi : (\Omega, a, b, c, d) \longrightarrow (H, 1-x, 1, \infty, 0)$ $(0 < x < 1)$
$x = x(\Omega, a, b, c, d)$. Then

$$C_0 = \lim_{\delta \to 0} C_\delta(\Omega, a, b, c, d) = F(x) = \frac{\int_0^x (s(1-s))^{-2/3}\, ds}{\int_0^1 (s(1-s))^{-2/3}\, ds}$$

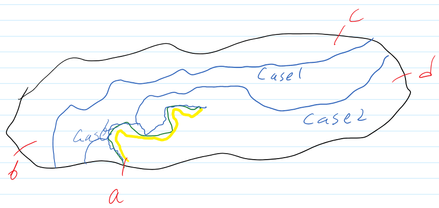




<u>Martingale property of C_δ:</u>

Let γ^δ be Exploration process from a to c in Ω.

Then $C_\delta(\Omega \setminus \gamma^\delta_{[0,t]}, \gamma^\delta(t), b, c, d) =$
$C_\delta(\Omega, a, b, c, d \mid \gamma^\delta_{[0,t]})$.



Indeed, two cases:

1) Blue crossing from (a,b) to (c,d) does not intersect $\gamma[0,t] \iff$ it is a crossing in $\Omega \setminus \gamma^\delta_{[0,t]}$ not intersecting $\gamma^\delta[0,t]$

2) Blue crossing in Ω intersects $\gamma[0,t] \iff$
$\exists$ blue crossing in $\Omega \setminus \gamma^\delta_{[0,t]}$ from left side of $\gamma^\delta_{[0,t]}$ to (c,d).

So $E^\delta\left(C_\delta\left(\Omega \setminus \gamma^\delta_{[0,t]}, \gamma^\delta(t), b, c, d\right)\right) =$
$C_\delta(\Omega, a, b, c, d)$.

Moreover, we can do it for any $s < t$ and domain $\Omega \setminus \gamma_{[0,s]}$ to get

$$E^\delta\left(C_\delta\left(\Omega \setminus \gamma^\delta_{[0,t]}, \gamma^\delta(t), b, c, d\right) \,\middle|\, \gamma^\delta[0,s]\right) =$$
$$C_\delta\left(\Omega \setminus \gamma^\delta_{[0,s]}, \gamma(s), b, c, d\right).$$

We want to take $\delta \to 0$ (so that everything depends on moduli). Nontrivial, since if $\mathbb{P}'$ is the law of a subsequential limit, then we need a $\mathbb{P}'$-typical curve.

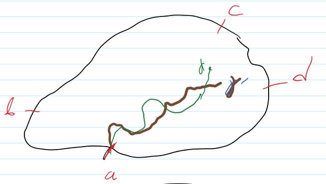
We'll need apriori estimate (for percolation!).

Lemma (Rigidity) Let γ be a Löwner curve in Ω

We'll need apriori estimate (for percolation!).

Lemma (Rigidity) Let γ be a Löwner curve in Ω starting at a, $\mathrm{dist}(\gamma, c) > \Delta$ (not entering some fixed neighborhood of c).
Then $\forall \varepsilon > 0\ \exists\ \delta(\varepsilon, \Delta),\ d(\gamma)$ such that
$\forall \gamma'$ – Löwner curve
$\mathrm{dist}(\gamma', \gamma) < d(\gamma)$, $\delta < \delta(\varepsilon, \Delta) \Rightarrow$

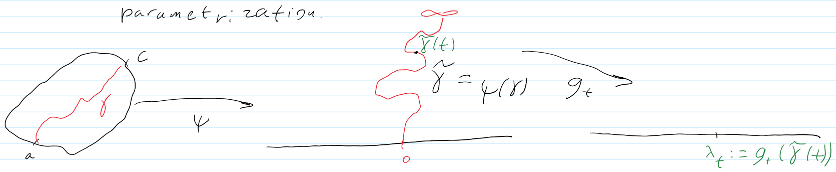
$$\left| C_\delta(\Omega \setminus \gamma'[0,t], \gamma'(t), b, c, d) - C_0(\Omega \setminus \gamma[0,t], \gamma(t), b, c, d) \right| < \varepsilon.$$



So $C_\delta \to C_0$ not only for γ, but in some neighborhood
True even when $C_0 = 0$ (b is absorbed) or $C_0 = 1$ (d is absorbed)

Let us now consider
$\varphi : (\Omega, a, c) \to (\mathbb{H}, 0, \infty)$ – conformal.
Parametrize Löwner curve $\varphi(\gamma)$ by half-plane capacity ($\mathrm{Hcap}(\varphi(\gamma[0,t])) = 2t$).
This is called Löwner parametrization of γ (depends on φ!).

Lemma Let γ be a Löwner curve in Löwner parametrization from a to c, $\Delta > 0$, $\mathrm{dist}(\gamma[0,t], c) > \Delta$.
Then $\forall \varepsilon > 0\ \exists\ \eta(\varepsilon, \Delta, \gamma)$, such that $\mathrm{dist}(\gamma', \gamma) < \eta \Rightarrow$
$|\gamma'(s) - \gamma(s)| < \varepsilon\ \ \forall s \le t$, where $\gamma'(s)$ is in Löwner parametrization.



Notation: $\tilde{\gamma} := \varphi(\gamma)$ – Löwner curve in $\mathbb{H}$, driven by λ_t.
Want: if $\tilde{\gamma}$ is a subsequential limit of $\tilde{\gamma}^\delta$, then λ_t has the law of $B(6t)$.

First: let $\delta \to 0$ in (*) (the martingale property).
Fix $\Delta > 0$, $\varepsilon > 0$. Let $\mathcal{L}_\Delta := \{$Löwner curves in Ω from a not intersecting Δ nbhd of $c\}$.
Choose countable collection of curves γ_n, so that

$\mathcal{L}_\Delta = \bigcup \underbrace{B(\gamma_n, d(\gamma_n, \varepsilon))}_{=: B_n}$, where $d(\gamma_n, \varepsilon)$ as in Rigidity Lemma,

i.e. $\mathrm{dist}(\gamma', \gamma_n) < d(\gamma_n, \varepsilon) \Rightarrow$

$$\Big| \underbrace{C_\delta(\Omega \setminus \gamma'[0,t], \gamma'_t, b, c, d)}_{=: K^\delta(\gamma')} - \underbrace{C_0(\Omega \setminus \gamma_n[0,t], \gamma_n(t), b, c, d)}_{=: K^0(\gamma_n)} \Big| < \varepsilon$$

Let P^0 be the law of a subsequential limit.

Fix $\alpha \ll 1$.

Now choose N disjoint sets:

$D_1 := B_1$, $D_2 = B_2 \setminus B_1$, ..., $D_N = B_N \setminus \bigcup_{j=1}^{N-1} B_j$; such that

$P^0(\mathcal{L}_\Delta \setminus \bigcup_{j=1}^N D_j) < \alpha$. Then for small δ_n, $P^{\delta_n}(\mathcal{L}_\Delta \setminus \bigcup_{j=1}^N D_j) < \alpha$, since $\overline{\lim}_{\delta_n \to 0} P^{\delta_n}(\mathcal{L}_\Delta \setminus \bigcup_{j=1}^N D_j) \leq P$ closed.

Then

$$\Big| E^\delta K^\delta(\gamma_t) - \sum_{j=1}^N P^\delta(D_j) K^\delta(\gamma_j(t)) \Big| \leq \alpha + \sum \sup_{\gamma' \in D_n} |K^\delta(\gamma') - K^\delta(\gamma_n)| \cdot P^\delta(D_n) \leq \alpha + \delta.$$

But the same is true for P^0:

$$\Big| E^0(K^0(\gamma_t)) - \sum_{j=1}^N P^0(D_j) K^0(\gamma_j(t)) \Big| \leq \alpha + \varepsilon.$$

Now $\Big| \sum P^\delta(D_j) K^\delta(\gamma_j) - \sum P^0(D_j) K^0(\gamma_j) \Big| \leq$

$$\sum_{j=1}^N \big| P^\delta(D_j) K^\delta(\gamma_j) - P^0(D_j) K^0(\gamma_j) \big| \; (\leq)$$

Observe: D_j is open, so $\underline{\lim}\, P^\delta(D_j) \geq P^0(D_j)$.

and $\sum P^\delta(D_j) - \sum P^0(D_j) < 2\alpha$.

so, for small δ,

$$\sum |P^\delta(D_j) - P^0(D_j)| < 3\alpha$$

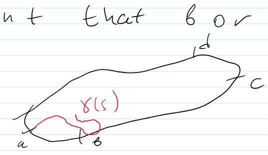
$(\leq)$ $\varepsilon + 3\alpha$ for small δ.

Now let $\delta \to 0$, $\varepsilon \to 0$, $\alpha \to 0$ to get that

$$E^0\big(C_0(\Omega \setminus \gamma[0,t], \gamma(t), b, c, d)\big) = C_0(\Omega, a, b, c, d).$$

We need also a version conditioned on $\gamma[0,s]$ for $s \leq t$.

The proof goes the same way with a little twist: we have to take into account the event that b or d can be swallowed before s.



$\chi_s := \{$neither b nor d swallowed before $s\}$.

Then $E^0\big(1_{\chi_t} C_0(\Omega \setminus \gamma[0,t], \gamma(t), b, c, d) / \gamma[0,s]\big) =$

$$1_{\chi_s} C_0(\Omega \setminus \gamma[0,s], \gamma(s), b, c, d).$$

We just established the following statement.

Statement. $1_{\chi_t} C_0(\Omega \setminus \gamma[0,t], \gamma(t), b, c, d)$ is a martingale with respect to P^0.

By Cardy formula:

$$k(\gamma[0,t]) = F(\underbrace{X(\gamma[0,t])}_{=: X_t}).$$

What is X_t?

$\tilde{a} := \psi(a) = 0$

$\tilde{b} := \psi(b)$

$\tilde{c} := \psi(c) = \infty$

$\tilde{d} := \psi(d)$

Let $$h_t(z) = \frac{g_t(z) - g_t(\tilde{d})}{g_t(\tilde{b}) - g_t(\tilde{d})}$$

Then $h_t(\psi(z))$ maps $(\Omega, \gamma(t), b, c, d)$ to $(\mathbb{H}, 1-X_t, 1, \infty, 0)$.

So $$X_t = 1 - \frac{\lambda_t - g_t(\tilde{d})}{g_t(\tilde{b}) - g_t(\tilde{d})} = \frac{g_t(\tilde{b}) - \lambda_t}{g_t(\tilde{b}) - g_t(\tilde{d})}$$

So $F\left(\frac{g_t(\tilde{b}) - \lambda_t}{g_t(\tilde{b}) - g_t(\tilde{d})}\right) 1_{\chi_t}$ is a martingale.

Let us ignore 1_{χ_t} for simplicity (it can be estimated).

So we have

$$E\left(F\left(\frac{g_t(\tilde{b}) - \lambda_t}{g_t(\tilde{b}) - g_t(\tilde{d})}\right) \Big| \gamma[0,s]\right) = F\left(\frac{g_s(\tilde{b}) - \lambda_s}{g_s(\tilde{b}) - g_s(\tilde{d})}\right)$$

Now fix s, t, take $\tilde{d} = -2\tilde{b}$ (we can vary b and d!).

Then, for large $\tilde{b}$

$$F\left(\frac{g_t(\tilde{b}) - \lambda_t}{g_t(\tilde{b}) - g_t(\tilde{d})}\right) = F\left(\frac{\tilde{b} - \lambda_t + \frac{2t}{\tilde{b}} + O\left(\frac{1}{\tilde{b}^2}\right)}{\tilde{b} + \frac{2t}{\tilde{b}} + 2\tilde{b} - \frac{t}{\tilde{b}} + O\left(\frac{1}{\tilde{b}^2}\right)}\right) =$$

$$F\left(\frac{1}{3}\right) - \frac{\lambda_t}{3} F'\left(\frac{1}{3}\right)\frac{1}{\tilde{b}} - \frac{t}{3} F'\left(\frac{1}{3}\right)\frac{1}{\tilde{b}^2} + \frac{\lambda_t^2}{18} F''\left(\frac{1}{3}\right) \frac{1}{\tilde{b}^2} + O\left(\frac{1}{\tilde{b}^3}\right) =:$$

$$A - \frac{B}{\tilde{b}}\lambda_t - \frac{C}{\tilde{b}^2}(\lambda_t^2 - 6t) + O\left(\frac{1}{\tilde{b}^3}\right)$$

Use martingale property

$$E(\lambda_t \mid \gamma[0,s]) = \lambda_s$$

$$E(\lambda_t^2 - 6t \mid \gamma[0,s]) = \lambda_s^2 - 6s$$

So λ_t – continuous time martingale,

$\lambda_t^2 - 6t$ – also martingale, so

$$\langle \lambda_t, \lambda_t \rangle = 6t \implies \lambda_t \overset{law}{=} B(6t)$$

Technical detail: need to bound $E(O(\frac{1}{\delta^3}))$

depends on λ_t!

Lemma 1) $P^0(\lambda_t > n) \le C_1 \exp(-C_2 \frac{n}{\sqrt{t}})$

2) $P(\sup_{s \le t} |\tilde{\gamma}_s| > n) \le C_1 \exp(-C_2 \frac{n}{\sqrt{t}})$

We'll need an auxiliary estimate:

Statement. 1) $\mathrm{Im}\, \tilde{\gamma}(t) \le 2\sqrt{t}$.

2) $\sup_{s \le t} |\tilde{\gamma}(s)| \ge \frac{|\lambda_t|}{4}$.

Proof of statement.

$\partial_t \mathrm{Im}\, g_t(z) = -2 \frac{\mathrm{Im}\, g_t(z)}{|g_t(z) - \lambda_t|^2} \ge \frac{-2}{\mathrm{Im}\, g_t(z)}$

so $(\mathrm{Im}\, g_t(z))^2 \ge (\mathrm{Im}\, z)^2 - 4t$. Take $z = \tilde{\gamma}(t)$, $\mathrm{Im}\, g_t(\tilde{\gamma}_t) = 0 \Rightarrow$

$0 \ge (\mathrm{Im}\, \tilde{\gamma}(t))^2 - 4t \Rightarrow \mathrm{Im}\, \tilde{\gamma}(t) \le 2\sqrt{t}$.

let $R_t = \sup_{s \le t} |\tilde{\gamma}(s)|$.

Then the corresponding $\tilde{K}_t \subset R_t \mathbb{D} \cap \mathbb{H}$.

So if $g(z) := z + \frac{R_t^2}{z}$, $g : \mathbb{H} \setminus R_t \mathbb{D} \to \mathbb{H}$,

Then $g_t \succ g$, so, in particular

$\forall |x| > R_t$, $x \in \mathbb{R}$, $|g_t(x)| \le |g(x)| = |x + \frac{R_t^2}{x}|$

On $\mathbb{R} \setminus K_t$, $|g_t(x) - x| \le R_t$

On K_t: $|g_t(x)| \le 2R_t \Rightarrow |g_t(x) - x| \le 3R_t$.

So, by Maximum principle, $|g_t(z) - z| \le 3R_t$.

Let $z \to \gamma(t)$, get $|\lambda_t - \gamma_t| \le 3R_t \Rightarrow$

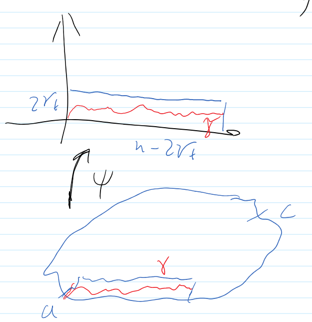
$4 \sup_{s \le t} |\gamma_s| \ge |\lambda_t|$ ■

Proof of Lemma.

By statement, 2) $\Rightarrow$ 1).

Also, by statement,

$\sup |\tilde{\gamma}_s| > n \Rightarrow \tilde{\gamma}[0,t]$ is a crossing of rectangle of modulus $\frac{n - 2\sqrt{t}}{2\sqrt{t}}$:



$\{0 \le x \le n - 2\sqrt{t};\ 0 \le y \le 2\sqrt{t}\}$

or $\{2\sqrt{t} - n \le x \le 0,\ 0 \le y \le 2\sqrt{t}\}$

$(\mathrm{Im}\, \gamma(s) \le 2\sqrt{t})$.

Which means that γ crosses a conformal rectangle of the same modulus. By Cardy formula, this probability is $\approx e^{-\frac{1}{3}\frac{n}{2\pi t\pi}}$